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Optimization of cylindrical shells with compliant cores

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Abstract

Thin-walled, cylindrical structures are found extensively in both engineering components and in nature. The weight to load bearing ratio is a critical element of design of such structures in a variety of engineering applications, including space shuttle fuel tanks, aircraft fuselages, and offshore oil platforms. In nature, thin-walled cylindrical structures are often supported by a honeycomb- or foam-like cellular core, as for example, in plant stems, porcupine quills, or hedgehog spines. Previous studies have suggested that a compliant core increases the buckling resistance of a cylindrical shell over that of a hollow cylinder of the same weight. In this paper, we extend the linear-elastic buckling theory by coupling it with basic plasticity theory to provide a more comprehensive analysis of isotropic, cylindrical shells with compliant cores. We examine the optimal design of a thin-walled cylinder with a compliant core, of given radius and specified materials, for a prescribed load bearing capacity in axial compression. The analysis gives the values of the shell thickness, the core thickness, and the core density that maximize the load bearing capacity of the shell with a compliant core over an equivalent weight hollow shell. The analysis also identifies the optimum ratio of the core modulus to the shell modulus and is supported by a Lagrangian optimization technique. The analysis further discusses the selection of materials in the design of a cylinder with a compliant core, identifying the most suitable material combinations. The performance of a cylinder with a compliant core is compared with competing designs (optimized hat-stiffened shell and optimized sandwich-wall shell). Finally, the challenges associated with achieving the optimal design in practice are discussed, and the potential for practical implementation is explored.

Keywords: Biomimetic; Buckling; Cellular; Core; Cylinder; Shell

1. Introduction

Thin-walled, cylindrical structures are found extensively in both engineering components and in nature. Typical shell radius to thickness ratios for a variety of cylindrical engineering structures can be seen in Fig. 1; in some applications, such as space shuttle fuel tanks, aircraft fuselages, and offshore oil platforms, the weight to load bearing ratio is an essential element of design. In nature, plant stems, animal quills, and

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Nomenclature

a	radius to mid-plane of thickness
E	Young's modulus of the shell material
E_c	Young's modulus of the core material
E_s	Young's modulus of the solid material the compliant cellular core is comprised of
F	minimum weight auxiliary function
f_1	critical buckling stress parameter
L	length of the cylinder
N	uniaxial compression load per unit circumferential length
P	specified required axial load for cylinder with compliant cellular core
P_C^b	axial compressive buckling failure load of cylinder with compliant cellular core
P_C^m	axial compressive material failure load of cylinder with compliant cellular core
P_H	axial compressive elastic buckling failure load of the hollow cylinder
t	thickness of the shell of the cylinder with compliant cellular core
t_c	thickness of the compliant cellular core
t_{eq}	equivalent thickness of hollow cylinder
w	total weight of cylinder with core
α	core modulus to shell modulus ratio $[E_c/E]$
α_o	numerical coefficient for equation relating modulus ratio to density ratio
β_o	numerical coefficient approximately equal to 0.58
λ_{cr}	buckling wavelength parameter minimizing N_x
η	core density ratio to shell density ratio $[\rho_c/\rho]$
v	Poisson's ratio of shell material
v_c	Poisson's ratio of core material
ρ	density of the shell material
ρ_c	density of the core material
ρ_s	density of the solid material the core is comprised of
σ_o	theoretical buckling stress in uniaxial compression of a hollow cylinder
σ_{cr}	axisymmetric buckling stress of shell with a core under axial compression
σ_f	failure stress of material

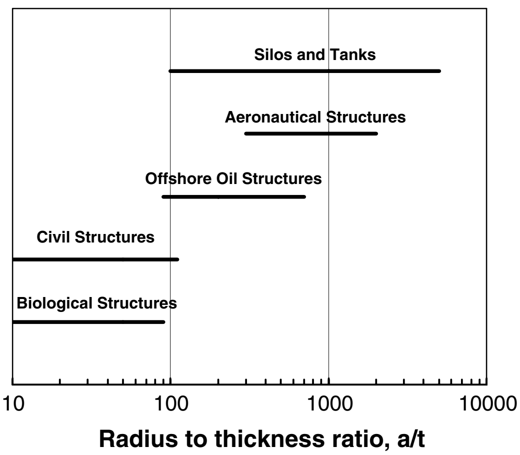

Fig. 1. Radius to thickness ratio a/t for typical engineering cylindrical structures (after Karam and Gibson, 1995a).

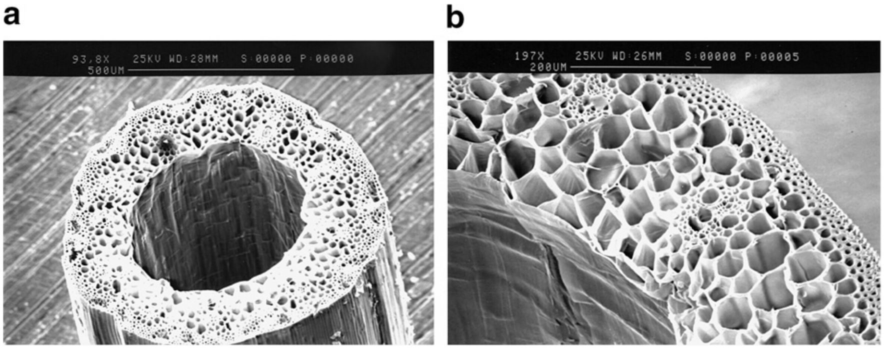

Fig. 2. Micrographs of natural shell structure with compliant core; cross section of grass stem (*Elytrigia repens*) with a foam-like core.

bird feather rachis have a dense cylindrical shell supported by a honeycomb- or foam-like cellular core that increases the resistance to buckling (Fig. 2).

Previous studies of elastic buckling have suggested a thin-walled, cylindrical shell, supported by a compliant core can achieve a higher buckling load than an equivalent hollow shell of the same weight and radius both for uniaxial compression and pure bending. The uniaxial load case has been solved using differential equations for equilibrium of the shell, modified to account for the spring constant of the compliant core (Seide, 1962) and by use of the stress functions (Yao, 1962). Both methods are applicable for thin-walled cylinders with a core modulus lower than that of the shell. Karam and Gibson (1995a) further analyzed the elastic buckling of a thin, isotropic cylindrical shell with a compliant elastic core to develop a simplified analysis for axisymmetric buckling in uniaxial compression and for local buckling in bending. All three analyses give similar results (Karam and Gibson, 1995b).

Previous experimental results have supported these analyses. Karam and Gibson (1995b) conducted experiments on the elastic buckling of thin-walled cylinders with elastic cores under uniaxial compression. The experimental results consistently demonstrated thin-walled shells with elastic cores outperformed the equivalent hollow counterparts, but the experimental results for the buckling load varied widely with respect to the expected values. The results gave buckling loads ranging from 30% to 90% of the predicted values with typical results on the order of 70% of the theoretical values.

In this paper, we extend the linear-elastic buckling theory from Karam and Gibson (1995a) by coupling it with basic plasticity theory to provide a more comprehensive analysis of isotropic, cylindrical shells with compliant cores. The goal of this paper is to examine the optimum design of thin-walled, cylindrical shells with compliant cores subjected to uniaxial compression. The analysis also examines the improvement in load carrying capacity of an optimized cylinder with a compliant core over that of competing designs (an equivalent hollow cylinder with equal weight and radius, the optimized hat-stiffened design, and the optimized sandwich design). The optimized design for the cylinder with a compliant core presented here shows significant improvement over all competing designs. This analysis further examines the feasibility of implementing compliant cores in thin-walled engineering structures where the weight to load bearing ratio is a critical element of design. The material and structural design constraints for introducing this design into engineering structures are discussed, and the most advantageous engineering materials are presented. Based on the constraints associated with thin-walled shells with compliant cores, recommendations are developed and the potential for implementation into select engineering structures is discussed.

2. Analysis

This analysis describes the optimum design of a thin-walled, cylindrical shell with a compliant core loaded in uniaxial compression. The analysis assumes the radius of the cylinder, length of the cylinder and the materials of the shell and the core are given. The required axial load capacity of the cylinder with the compliant core, P, is also specified. The analysis finds the values of the shell thickness, the core thickness, and the core density that maximize the load to weight ratio of the cylinder with the compliant core.

2.1. Design configuration

A thin-walled shell with a compliant core, has an overall radius a, length L, outer shell thickness t, inner core thickness t_c, and weight w. It is compared with an equivalent hollow cylinder of identical radius a, identical length L, identical weight w, and wall thickness t_{eq} (Fig. 3). The outer shell of the cylinder with the compliant core and the hollow cylinder are made of the same isotropic material, with density ρ, Young's modulus E, and Poisson's ratio ν. Similarly, the core has density ρ_c, Young's modulus E_c, and Poisson's ratio ν_c.

We limit our analysis to shells with large a/t ratios. The materials under consideration are considered to behave linearly elastically up to the material failure, which we take to be deviation from the linear elasticity. For simplification, Poisson's ratio has been evaluated for all of the figures using $\nu = \nu_c = 0.3$.

2.2. Literature review

This analysis takes into account both elastic buckling and material failure. In uniaxial compression, material failure occurs at a critical stress equal to the material's failure stress (e.g. for a metal, the yield stress). Using linear-elastic theory and the theory of small-deflections, elastic buckling of a thin-walled, hollow, cylindrical shell of radius a and wall thickness t_{eq}, made of an isotropic material of Young's modulus E and Poisson's ratio ν takes place at a critical stress of Timoshenko and Gere (1961)

$$\sigma_o = \frac{Et_{eq}}{a\sqrt{3(1-\nu^2)}} \tag{1}$$

for any assumed buckling mode. Elastic buckling of a thin-walled cylinder with a compliant core in uniaxial compression takes place at a critical stress of Karam and Gibson (1995a)

$$\sigma_{cr} = \sqrt{3(1-\nu^2)}\sigma_o f_1, \tag{2}$$

where σ_o is the buckling stress for the hollow cylinder described in Eq. (1) with t_{eq} replaced by t for the cylinder with the compliant core and

$$f_1 = \frac{1}{12(1-\nu^2)}\frac{a/t}{(\lambda_{cr}/t)^2} + \frac{(\lambda_{cr}/t)^2}{a/t} + \frac{2\alpha}{(3-\nu_c)(1+\nu_c)}(\lambda_{cr}/t)(a/t). \tag{3}$$

This elastic buckling analysis is based on wrinkling of a flat sheet on an elastic foundation (Allen, 1969). As shown in Fig. 4 the buckling wavelength parameter, λ_{cr}, is independent of the radius to thickness ratio, a/t for

$$\left(\frac{E_c}{E}\right) > \left(4\frac{t}{a}\right)^{\frac{3}{2}}. \tag{4}$$

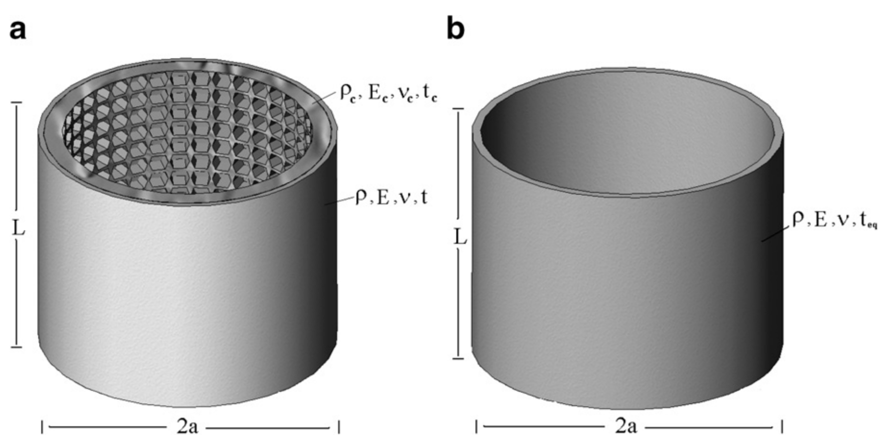

Fig. 3. (a) A thin-walled cylindrical shell with a honeycomb core. (b) An equivalent thin-walled hollow cylindrical shell.

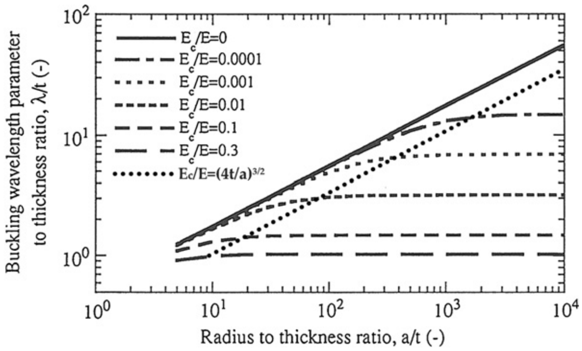

Fig. 4. The normalized axisymmetric buckling wavelength parameter λ_{cr}/t for a cylindrical shell plotted against radius to thickness ratio a/t for various values of E_c/E. Each curve can be approximated by a bilinear relationship. The E_c/E equation represents the minimum modulus ratio for the compliant core to act as an elastic foundation (after Karam and Gibson, 1995a).

We take this to be the minimum modulus ratio for this analysis. For this case, the normalized axisymmetric buckling wavelength parameter λ_{cr}/t is given by Allen (1969)

$$\left(\frac{\lambda_{cr}}{t}\right) = \left[\frac{(3-\nu_c)(1+\nu_c)}{12(1-\nu^2)}\right]^{\frac{1}{3}}\left[\frac{E}{E_c}\right]^{\frac{1}{3}}, \tag{5}$$

where λ_{cr} is the buckling half wavelength divided by π. (Note that this does not reduce to the solution for the empty hollow shell for $E_c/E = 0$.)

The stresses within the compliant core decay radially such that they become negligible at a depth into the core of 1.6 times the buckling half wavelength or $5\lambda_{cr}$. The thickness of the compliant core t_c is taken to be this depth. The length of the shell L is also assumed to be at least several times the buckling half wavelength.

The core modulus to shell modulus ratio $\alpha = E_c/E$, is assumed to behave according to the following relationships for both foam and honeycomb cores (Gibson and Ashby, 1997):

$$\left(\frac{E_c}{E}\right) = C_1\left(\frac{\rho_c}{\rho_s}\right)\left(\frac{E_s}{E}\right) \quad \text{Honeycomb}, \tag{6}$$

$$\left(\frac{E_c}{E}\right) = C_2\left(\frac{\rho_c}{\rho_s}\right)^2\left(\frac{E_s}{E}\right) \quad \text{Open-cell Foam}, \tag{7}$$

where E_s and ρ_s are the modulus and the density of the solid material comprising the core and C_1 and C_2 are constants equal to 1.

The thickness of an equivalent hollow cylinder t_{eq}, of equal diameter and weight to that of the cylinder with a compliant core, is then calculated to be (Karam and Gibson, 1995a):

$$t_{eq} = t\left(1 + \frac{t_c}{2t}\frac{\rho_c}{\rho}\left[2 - \frac{t_c}{a}\right]\right). \tag{8}$$

2.3. Failure load analysis

For thin-walled shells with $a/t > 100$, the variable f_1 in Eq. (2) for the critical buckling stress of a thin-walled cylinder with a compliant core can be approximated to within 5% of the exact value by (after Cheng, 1994)

$$f_1 = \frac{(a/t)}{4(1-\nu^2)\left(\frac{\lambda_{cr}}{t}\right)^2}. \tag{9}$$

Substituting f_1 into the critical buckling stress equation then gives (Cheng, 1994)

$$\sigma_{\mathrm{cr}} = \frac{E}{4(1-\nu^2)\left(\frac{\lambda_{\mathrm{cr}}}{t}\right)^2}. \tag{10}$$

We note that for the cylinder with the compliant core, the load is assumed to be entirely supported by the shell. This is justified by the fact the core modulus in the plane of the load is negligible for both honeycomb and foam cores. Therefore, the compliant core is assumed to behave like an elastic foundation for this analysis. The resulting critical failure loads for the shell with the compliant core are given by

$$P^{\mathrm{b}}_{[\mathrm{c}]} = \frac{\pi a t E}{2(1-\nu^2)\left(\frac{\lambda_{\mathrm{cr}}}{t}\right)^2} \quad \text{Elastic Buckling}, \tag{11}$$

$$P^{\mathrm{m}}_{[\mathrm{c}]} = 2\pi a t \sigma_{\mathrm{f}} \quad \text{Material Failure}. \tag{12}$$

In this analysis, it is important to incorporate the possibility of material failure. If there is material failure of the hollow cylinder, then the axial load carrying capacity of the cylinder with the compliant core will always be less than that of the hollow cylinder since the shell thickness for the cylinder with the compliant core is always less than that of the corresponding hollow cylinder.

Here, we only examine the two remaining failure scenarios where the hollow cylinder fails by elastic buckling and the corresponding cylinder with the compliant core fails by either material failure or elastic buckling. For the hollow cylinder to fail by elastic buckling

$$\frac{a}{t_{\mathrm{eq}}} \geqslant \frac{E}{\sqrt{3(1-\nu^2)}\sigma_{\mathrm{f}}}. \tag{13}$$

At lower ratios of a/t_{eq} material failure occurs.

Assuming that the hollow cylinder is designed to fail in buckling, we now analyze the two possible failure modes of the cylinder with a compliant core, buckling and material failure. The transition between the buckling and material failure is found to depend only on the material properties of the shell and the core because the ratio of a/t is assumed to be large, satisfying the constraint for the equivalent hollow shell given by Eq. (13). The transition for a cylinder with the compliant core can be determined by substituting Eq. (5) into Eq. (10) and setting the buckling failure stress equal to the material failure stress, giving the value of $(E_{\mathrm{c}}/E)_{\mathrm{transition}}$ to be

$$\left(\frac{E_{\mathrm{c}}}{E}\right)_{\mathrm{transition}} = \left(\frac{2}{3}\right)(1+\nu_{\mathrm{c}})(3-\nu_{\mathrm{c}})(\sqrt{1-\nu^2})\left(\frac{\sigma_{\mathrm{f}}}{E}\right)^{3/2}. \tag{14}$$

As the ratio of the stiffness of the core to the stiffness of the shell is increased, the cylinder with the compliant core transitions to the material failure region. Combining Eq. (1) for buckling failure of the hollow cylinder with Eq. (12) for material failure of the cylinder with a compliant core gives the ratio of the axial capacity of the cylinder with a compliant core to that of the corresponding hollow cylinder, which is referred to as the load ratio, $P^{\mathrm{m}}_{[\mathrm{C}]}/P_{[\mathrm{H}]}$

$$\left(\frac{P^{\mathrm{m}}_{\mathrm{C}}}{P_{\mathrm{H}}}\right) = \frac{\sigma_{\mathrm{f}} a t\sqrt{3(1-\nu^2)}}{E t_{\mathrm{eq}}^2} \tag{15}$$

$$\text{for} \quad \left(\frac{E_{\mathrm{c}}}{E}\right) > \left(\frac{2}{3}\right)(1+\nu_{\mathrm{c}})(3-\nu_{\mathrm{c}})(\sqrt{1-\nu^2})\left(\frac{\sigma_{\mathrm{f}}}{E}\right)^{3/2}.$$

As the modulus ratio decreases below the value given by Eq. (14), the cylinder with a compliant core transitions to the buckling failure region. Using Eq. (1) for buckling failure of the hollow cylinder and Eq. (11) for buckling failure of the cylinder with the compliant core, the corresponding load ratio $P^{\mathrm{b}}_{[\mathrm{C}]}/P_{[\mathrm{H}]}$ is given by

$$\left(\frac{P^{\mathrm{b}}_{\mathrm{C}}}{P_{\mathrm{H}}}\right) = \frac{2.27 a t (1-\nu^2)^{1/6}}{\left[(3-\nu_{\mathrm{c}})(1+\nu_{\mathrm{c}})\left(\frac{E}{E_{\mathrm{c}}}\right)\right]^{2/3} t_{\mathrm{eq}}^2} \tag{16}$$

$$\text{for} \left(\frac{E_{\mathrm{c}}}{E}\right) < \left(\frac{2}{3}\right)(1+\nu_{\mathrm{c}})(3-\nu_{\mathrm{c}})(\sqrt{1-\nu^2})\left(\frac{\sigma_{\mathrm{f}}}{E}\right)^{3/2}.$$

2.4. Optimization of configuration based on E_c/E

The design problem presented contains non-linear equations with inequality constraints, requiring an analytical optimization for the optimum design of a cylinder with compliant core to be a function of only one variable. All of the design variables for the cylinder with the compliant core can be written in terms of the ratio of the modulus of the core relative to that of the shell, E_c/E. The core density, ρ_c, is related to E_c/E, through Eq. (6) or Eq. (7). The core thickness, $_c$, at which the stresses within the core became negligible is $5\lambda_{cr}$, and λ_{cr} is related to E_c/E through Eq. (5). The hollow shell thickness, t_{eq}, can then be related to E_c/E through Eq. (8), since t can be found from Eq. (11) or Eq. (12) for a given axial load bearing capacity of the cylinder with the compliant core. For the design problem presented the maximum improvement in the axial load ratio $P_{[C]}/P_{[H]}$, is found by determining the optimal value of the modulus ratio (Fig. 5). The range over which this model is valid is demonstrated in Fig. 5. The minimum value of E_c/E is given by Eq. (4). The maximum value is taken to be $E_c/E = 0.1$. This value is chosen to reflect the wrinkling analysis of Allen (1969), which assumes a compliant core acting as an elastic foundation beneath a stiffer skin, as well as the assumption that the core carries no axial in-plane load.

For the cylinder with a compliant core, failing in the material failure region, the improvement in the load ratio is given by Eq. (15). Taking the partial derivative of Eq. (15) (with t and t_{eq} in terms of E_c/E) with respect

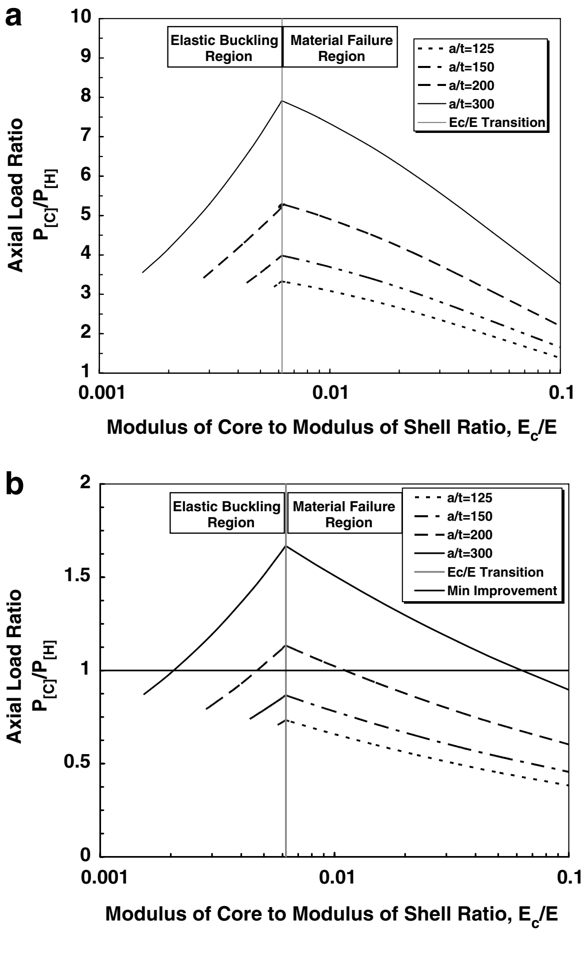

Fig. 5. The ratio of the failure load in axial compression of a glass fiber-epoxy composite shell with a compliant core to an equivalent hollow shell plotted against the ratio of core modulus to shell modulus. $E_c/E_{transition}$ for E-glass is .0062. (a) Honeycomb core; (b) foam core.

to E_c/E and setting it equal to zero, gives the value of E_c/E, which maximizes the improvement in the load ratio. For radius to thickness ratios satisfying Eq. (13), the optimal value of E_c/E approaches zero, but the minimum value in the material failure region is the transition value $(E_c/E)_{\text{transition}}$ given in Eq. (14). Therefore, $(E_c/E)_{\text{transition}}$ is the limiting value, and the optimal value for the material failure region. This analysis assumes the shell and the core are made of the same material, but a parametric study, varying the density of the core material, reveals the transition modulus ratio is the optimal modulus ratio for all engineering materials.

In the buckling failure region the improvement in the load ratio is given by Eq. (16). Maximization of Eq. (16) with respect to E_c/E shows that the ideal value for shells with radius to thickness ratios satisfying the value given by Eq. (13) is greater than the transition value $(E_c/E)_{\text{transition}}$. In the buckling region, the transition modulus ratio is again found to be the limiting value, and therefore the optimal value. As in the material failure region, a parametric study, varying the density of the core material, reveals the analysis is valid for all engineering materials.

The optimal value for E_c/E, which maximizes the load ratio $P_{[C]}/P_{[H]}$, for both the material and buckling failure regions is the transition E_c/E given by

$$\left(\frac{E_c}{E}\right)_{\text{optimum}} = \frac{2(1+\nu_c)(3-\nu_c)\sqrt{1-\nu^2}}{3}\left(\frac{\sigma_f}{E}\right)^{3/2}. \tag{17}$$

This analysis demonstrates that optimization of the improvement in the load ratio in axial compression is independent of whether or not the core is a honeycomb structure or a foam structure. It is also independent of the radius to thickness ratio a/t, because the a/t ratio is incorporated into the analysis through the constraint given by Eq. (13). This observation is demonstrated in Fig. 5 where the load ratio is plotted against the core modulus to shell modulus ratio E_c/E, for a variety of a/t ratios when the shell and core are composed of an E-glass fiber composite, assumed to be isotropic.

The improvement in the load ratio for a cylindrical shell with honeycomb core over an equivalent hollow shell is significantly greater than that for a cylindrical shell with foam core for any given a/t ratio. This was evident throughout the study independent of the materials comprising the shell and the core; therefore, the focus of this analysis is primarily on cylindrical shells with compliant honeycomb cores.

The optimization described is valid for all isotropic structural material combinations. Fig. 6 demonstrates the optimization for three common engineering materials: the optimum value of E_c/E is always given by $(E_c/E)_{\text{transition}}$ (Eq. (14)). Fig. 6 also shows that the improvement in the load ratio for any modulus ratio in the buckling region is independent of the material (Eq. (16)). Table 1 gives the material properties used to generate the figures. In order to compare the cylinder with the compliant core to a hollow cylinder of equal weight, the density of the core is required. For consistency Fig. 6 assumes the shell and the core are composed of the same material.

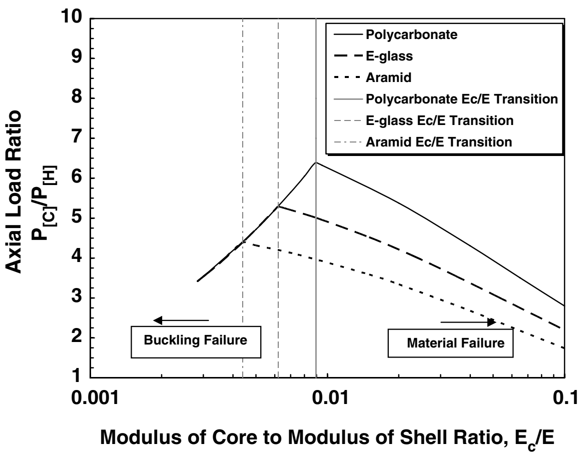

Fig. 6. The ratio of elastic buckling load for uniaxial compression of a honeycomb shell with a compliant core to an equivalent hollow shell plotted against the ratio of core modulus to shell modulus. All cylinders have radius to thickness ratios, $a/t = 200$.

Table 1
Material properties used in generation of figures, selected as representative values of typical properties

Engineering material	Modulus, E (GPa)	Material failure stress, σ_f (MPa)	Density, ρ (g/cm^3)
Polycarbonate	2.6	66	1.2
E-glass	38	750	1.8
Aramid	83	1300	1.4

Sources: Data supplied by manufacturers and Shackelford (2000).

Based on the optimal modulus ratio, the value of the relative core density ρ_c can then be determined, assuming honeycomb and foam cores behave according to Eq. (6) and Eq. (7), respectively. Therefore, the optimum value of the relative core density ρ_c, is given by

$$\rho_c = \rho_s \frac{2(1+\nu_c)(3-\nu_c)\sqrt{1-\nu^2}}{3}\left(\frac{\sigma_f}{E_s}\right)\left(\frac{\sigma_f}{E}\right)^{1/2} \quad \text{Honeycomb,} \tag{18}$$

$$\rho_c = \rho_s \sqrt{\frac{2(1+\nu_c)(3-\nu_c)\sqrt{1-\nu^2}}{3}}\left(\frac{\sigma_f}{E_s}\right)^{1/2}\left(\frac{\sigma_f}{E}\right)^{1/4} \quad \text{Foam.} \tag{19}$$

Given that elastic buckling and material failure occur simultaneously, Eq. (12) can be rearranged to give the optimal value of the shell thickness that maximizes the load carrying capacity of the cylinder with the compliant core

$$t = \frac{P}{2\pi a \sigma_f}. \tag{20}$$

Substituting Eq. (17) into Eq. (5) gives the buckling wavelength parameter as

$$\frac{\lambda}{t} = \frac{1}{2}\left[\frac{E}{(1-\nu^2)\sigma_f}\right]^{1/2}. \tag{21}$$

Since the thickness of the compliant core t_c is given as 1.6 times the buckling half wavelength, or $5\lambda_{cr}$, the optimal core thickness is given as

$$t_c = \frac{5P}{4\pi a \sigma_f}\left[\frac{E}{(1-\nu^2)\sigma_f}\right]^{1/2}. \tag{22}$$

Eqs. (18)–(20), and (22) then give the core density, the shell thickness, and the core thickness that maximize the axial load ratio of a cylinder with a compliant honeycomb or foam core to an equivalent hollow cylinder for a prescribed axial load applied to the cylinder with a compliant core.

Optimization techniques are often based on the method of Lagrange multipliers. However, because Lagrangian optimization techniques are not valid for multi-variable, non-linear equations with inequality constraints, we utilize a technique of assuming multiple failure modes occur simultaneously to reduce inequality constraints into equality constraints. The auxiliary function, $F(t, t_c)$, characterizing the minimum weight design of the shell, is given by

$$F(t,t_c) = W - 2\pi L\left[at\rho + t_c\rho_c\left(a - \frac{t_c}{2}\right)\right] - \lambda_1[N - t\sigma_f] - \lambda_2\left[N - \frac{Et}{4(1-\nu^2)\left(\frac{\lambda}{t}\right)^2}\right], \tag{23}$$

where Eq. (5) can be substituted for λ_{cr}/t and Eq. (6) (for a honeycomb core) or Eq. (7) (for a foam core) can be rearranged to replace ρ_c. The optimization given in Eq. (23) is based on optimization of the thickness of the face t, the thickness of the core t_c, and the core density, ρ_c. We assume the thickness of the core is still equal to $5\lambda_{cr}$, which is a function of the shell thickness and the core modulus as seen in Eq. (5).

Assuming both constraints are active and buckling and material failure occur simultaneously, a Lagrangian optimization verifies the previous analysis. Substituting the partial derivative of Eq. (23) with respect to λ_1 into the partial derivative of Eq. (23) with respect to λ_2 and solving for the modulus ratio gives the optimal value in Eq. (17). Further, we prove the assumption that the optimum design occurs when both failure modes are active by examining each constraint separately. Assuming the only active constraint is λ_1, corresponding to the material failure region, the Lagrangian optimization shows the minimal weight of the cylinder with a compliant core occurs for a shell thickness according to Eq. (22) coupled with the largest core thickness achievable. As seen by substituting $t_c = 5\lambda_c$ into Eq. (5) the largest core thickness practicable would drive the optimal solution for a cylinder with a compliant core in the material failure region toward the buckling region. A similar result is found if the only active constraint is λ_2, corresponding to the buckling region. In this region the optimal value for the shell thickness is as large as attainable while the optimal value for the core thickness is as small as achievable. Both the optimal shell thickness and core thickness drive the optimal cylinder with a compliant core in the buckling region toward the material failure region. This Lagrangian technique verifies the optimization of the shell occurs when the buckling and material failure modes occur simultaneously, supporting the previous optimization method outlined in this paper.

2.5. Material selection

The feasibility of this design has been examined for a wide variety of materials. Based on the criteria that the equivalent hollow cylinder must fail by elastic buckling, Eq. (13) indicates materials with low E/σ_f ratios are ideal for the shell of the cylinder with a compliant core. The modulus ratio must satisfy Eq. (4) for Allen's wrinkling solution to be applicable and ideally would approach the optimal modulus given by Eq. (17). This also suggests that materials with low ratios of E/σ_f are ideal for minimizing the required radius to thickness ratio. Among structural engineering materials, both composites and polymers, with E/σ_f ratios in the range of 40–60, are candidates for improved performance with the addition of an appropriately designed honeycomb core. For metals, however, with E/σ_f ratios in the range of 150–500, achieving the optimal modulus ratio (Eq. (17)) while satisfying the minimum modulus ratio (Eq. (4)) requires large a/t ratios.

Fig. 7 plots P_C/P_H for shells made of typical engineering materials that could be used for thin-walled shells, assuming that the core is are made of the same material as the shell and are optimized based on the modulus ratio (Eq. (17)). As a/t increases, the benefit of using the compliant core increases. Fig. 8 plots P_C/P_H for a glass-epoxy fiber composite shell, for a variety of core materials.

Figs. 7 and 8 are based on the assumption the ideal core modulus to shell modulus ratio could be achieved. As previously discussed ideal values are difficult to achieve, but combinations of selected materials have proven to approach the optimal modulus ratios. Because no intrinsic relationship exists between the modulus of a given material and the density of that material, a relationship between modulus ratio and the ratio of the core density to the shell density cannot be established analytically. Therefore, an analytical optimization of materials comprising the core and the shell cannot be generalized; however, material selection can still be utilized to increase the efficiency of the design. Based upon Eq. (6) and Eq. (7), if E_c/E is held constant and E_s/E is increased, ρ_c/ρ must decrease. Therefore, the core density and thus, the weight can be reduced if the material composing the core is stiff compared to the material composing the shell for a given E_c/E. Ideally, a core material would be selected that has a relatively high Young's modulus with respect to the modulus of the shell material. Fig. 8 demonstrates this idealization, showing greater improvement in the load ratio for cylinders with compliant cores made of a material with higher modulus than the modulus of the material comprising the shell, $E_s \geqslant E$. The material properties can be found in Table 1. The cylinders with compliant cores in Fig. 8 are assumed to be optimized according to Eq. (17). In practice, the optimal value of E_c/E is often lower than the minimum achievable ρ_c/ρ_s, requiring the modulus of the core material E_s to be less than the modulus of the shell E, significantly reducing and often eliminating the advantage of the cylinder with a compliant core.

3. Design comparison

3.1. Design comparison—cellular-core, sandwich, and hat-stiffened designs

Engineering cylinders used in load bearing applications are typically supported by an internal-mesh supporting structure. An internal support composed of ring-stiffeners or longitudinal stringers increases the load

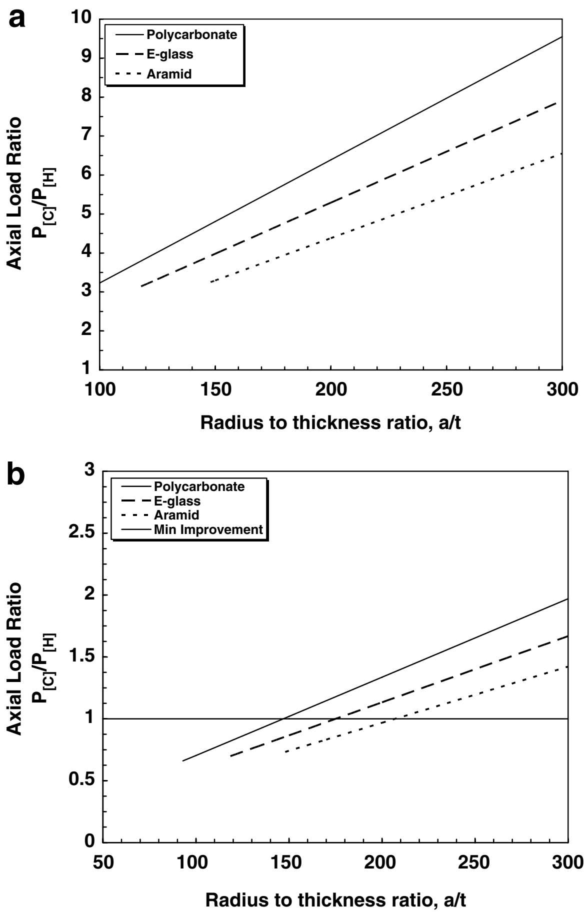

Fig. 7. The ratio of elastic buckling load for uniaxial loading of a cylinder with a compliant core to that of an equivalent hollow cylinder plotted against the ratio of shell radius to the shell thickness. (a) Honeycomb core; (b) foam core.

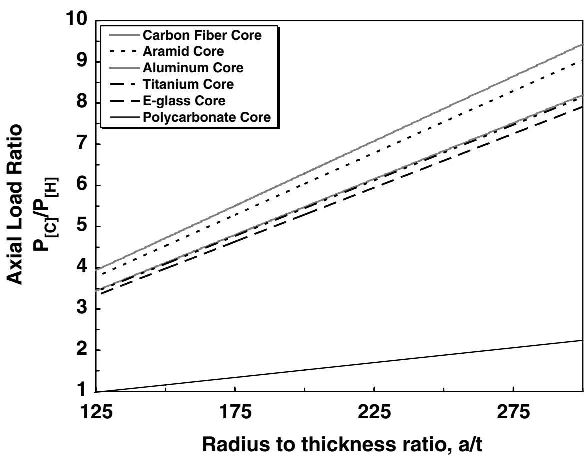

Fig. 8. The ratio of elastic buckling load for uniaxial loading of an optimized cylinder with E-glass fiber composite shell and a honeycomb compliant core to that of an equivalent hollow cylinder plotted against the ratio of shell radius to the shell thickness.

bearing capability of the cylinder and makes it more resistant to local buckling, primarily due to an increase in the resistance to material imperfections. In this analysis we refer to one of the most common support structures which is a specific type of longitudinal stiffener often referred to as a hat-stiffened structure as seen in Hutchinson and He (2000). It has been suggested these commonly used structures could be replaced by structures with a cellular core. There are two types of structures with cellular cores. The first is the type shown in Fig. 3, which is the focus of this paper and referred to as the cylinder with a compliant core design. This design uses a core, acting as an elastic foundation, to support one face plate. The second is referred to as a sandwich structure. This structure has a compliant core sandwiched between two face plates. A brief comparison of these designs demonstrates that the shell with a compliant core is superior to both the sandwich design and the hat-stiffened design.

Hutchinson used Lagrangian methods to optimize the sandwich design for a cylindrical shell. He performed two optimization methods of the auxiliary function, $F(t, t_c, \eta)$, characterizing the minimum weight of the shell (Hutchinson and He, 2000). In the first method he assumed the relative core density, ρ was specified and performed an optimization based on the thickness of the faces, t and the thickness of the core, t_c. In the second method, which he termed the *global optimum*, the relative core density, η was also treated as a variable in the minimization process along with t and t_c. Hutchinson's analysis assumed the modulus ratio, α is proportional to the square of the relative core density, η^2. This corresponds to a foam core as seen in Eq. (7). Using a Lagrangian technique, a similar analysis can be performed for a honeycomb core, assuming α is proportional to η as seen in Eq. (6). The corresponding auxiliary function becomes

$$F(t, t_c, \eta) = W - \lambda_1\left[N - \frac{2Ett_c/a}{\sqrt{1-\nu^2}}\left(1 - \frac{4\eta^{-1}t/a}{3\alpha_o\sqrt{1-\nu^2}}\right)\right] - \lambda_2[N - 2t\sigma_y] - \lambda_3[N - 2B_oE\alpha_o^{2/3}t\eta^{2/3}]. \tag{24}$$

Continuing with Hutchinson's optimization method for a sandwich shell, an equivalent optimized sandwich shell composed of a honeycomb core can be derived. A detailed analysis is included in Appendix A. For the first method where the relative core density, η is assumed to be given, the following Hutchinson constraint remains the same:

$$\frac{t}{a} = \frac{N}{2\sigma_o a}, \tag{25}$$

but the core thickness to radius constraint becomes

$$\frac{t_c}{a} = \frac{N}{ER}[a_1 t/a - a_2(t/a)^2\eta^{-1}]^{-1}. \tag{26}$$

For the second method where the relative core density, ρ is also treated as an optimization variable, optimization equations for three distinct failure regions can be determined. The constraint relations for the corresponding failure regions are as follows:

Overall buckling with face sheet yielding but no wrinkling

$$\text{(a) } \eta = \frac{a_2N}{a_1a\sigma_y}, \quad \text{(b) } \frac{t}{a} = \frac{a_1\eta}{2a_2}, \quad \text{(c) } \frac{t_c}{a} = \frac{N}{Ea}[a_1t/a - a_2(t/a)^2\eta^{-1}]^{-1}. \tag{27}$$

Overall buckling with face sheet wrinkling but no yielding

$$\text{(a) } 0 = \frac{4t - 5t_c\eta}{t_c} + \left(\frac{5a_2t/a}{[a_1 - a_2\eta^{-1}t/a]}\right)^*, \quad \text{(b) } \frac{t_c}{a} = \frac{N}{Ea}[a_1t/a - a_2(t/a)^2\eta^{-1}]^{-1}. \tag{28}$$

*Overall buckling with face sheet yielding and wrinkling**

$$\text{(a) } \eta = \left(\frac{\sigma_y}{E\beta_o\alpha_o^{2/3}}\right)^{3/2}, \quad \text{(b) } \frac{t}{a} = \frac{N}{2\sigma_o a}, \quad \text{(c) } \frac{t_c}{a} = \frac{N}{Ea}[a_1t/a - a_2(t/a)^2\eta^{-1}]^{-1}. \tag{29}$$

A more in-depth analysis, detailing the generation of the relations for a honeycomb core is provided in Appendix A. Using the constraint equations found in Eqs. (27)–(29), a plot of the weight index versus the load index for an optimized cylinder with honeycomb sandwich core can be developed as shown in Fig. 9.

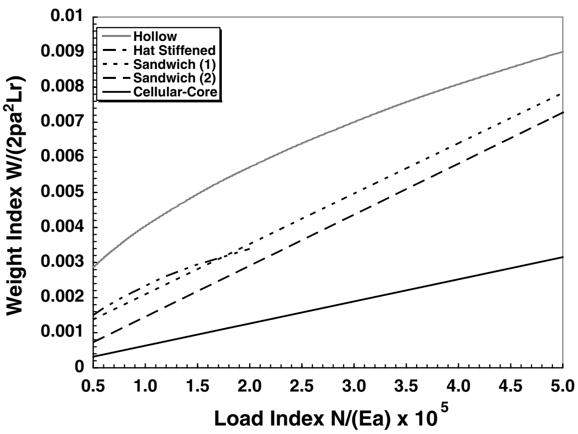

Fig. 9. The weight index as a function of the load index, comparing our cylinder with the compliant core to an equivalent hollow cylinder, an equivalent optimized hat-stiffened cylinder, and equivalent optimized sandwich-cored cylinders. (1) Sandwich design based on optimization of t and t_c with $\eta = 0.1$; (2) sandwich design based on optimization of t, t_c, and η. The hat-stiffened cylinder is assumed to be supported by rings spaced a distance equal to the shell radius (after Hutchinson and He, 2000). All materials are assumed to be glass-epoxy.

For each value of load index in Fig. 9, each design has been optimized to minimize its weight. The optimized, compliant cylinder with a honeycomb core shows dramatic improvement over the equivalent optimized honeycomb-sandwich cylinder, the[1] equivalent hat-stiffened cylinder, and the equivalent hollow cylinder.

4. Discussion

In nature, thin-walled cylindrical shells are commonly supported by a compliant cellular core. This analysis discusses the applicability of extending nature's model into thin-walled, engineering structures, such as space shuttle fuel tanks, aircraft fuselages, and offshore oil platforms. Currently, engineering cylinders used in load bearing applications are typically supported by an internal-mesh supporting structure. An internal support increases the load bearing capability of the cylinder and makes it less sensitive to defects and more resistant to local buckling. Recent improvements in manufacturing techniques have allowed for the cost effective introduction of cellular-core supports into commercial applications.

The analysis of a thin-walled shell with a compliant core presented discusses the optimal design for a given radius, prescribed materials, and specified axial load. The analysis is presented as a design guide, giving the values of the shell thickness, the core thickness, and the core density that maximize the load capacity of the cylinder with the compliant core relative to that of an equivalent hollow cylinder. The analysis also reveals a honeycomb core configuration is more effective than a foam core configuration. Theoretically, the optimized cylinder with a honeycomb, compliant core demonstrates substantial improvement in load bearing capability over the comparable designs discussed. However, achieving the optimal design, based on the modulus ratio, requires appropriate material selection. The thin-walled shells must be manufactured of a material which can be formed to a large radius to thickness ratio as governed by Eq. (13). In order to satisfy both the optimal modulus ratio and the constraint for the minimum modulus ratio given in Eq. (4) for a reasonable radius to thickness value, the shell must be composed of a material with a relatively large ratio of material failure stress to elastic modulus. Materials such as polymers and select composites have the greatest potential for practical applications. High strength alloys also prove to be practical for applications with very large radius to thickness ratios, but present a challenge to manufacture and test on the small scale. Theoretically, the

[1] The modified constraint equation for the "Overall buckling with wrinkling but no yielding" scenario cannot be generated analytically and must be solved numerically.

optimized shell with a compliant core demonstrates promise, but manufacturing this design on a small laboratory scale (less than 30 cm in diameter) presents numerous challenges.

Manufacturing and testing the optimally designed cylinder with compliant core was not successfully achieved due to challenges associated with the scale of testing. Specimens were found to deviate from the theoretical design due to numerous manufacturing constraints. Factors contributing to the deviation from the theoretical design include the weight of the epoxy required to bond the core to the shell, the increased sensitivity to the shell thickness tolerance, the increased sensitivity to imperfections, and the short buckling wavelength of the shell.

The analytical analysis does not account for the weight of the epoxy to bond the compliant core to the cylinders. On a small scale, the epoxy was found to increase the weight of the specimen on the order of 15%. The additional weight of the epoxy is considered dead weight and causes significant deviation from the theoretical design. Cylindrical shells manufactured on this scale were also found to have thickness tolerances on the order of 15% of the shell thicknesses. Because the cylinders with the compliant cores are designed with a thinner shell than the hollow cylinders, a greater deviation in the load bearing capability of the compliant core design is expected. Finally, the predicted buckling wavelength is linearly dependent on the shell thickness, which is relatively thin for the designs considered on a lab scale. On this scale, the buckling wavelength is found to be less than the cell-size of the honeycomb cores, demonstrating that the honeycomb is not acting as a continuum elastic foundation. The relatively small buckling wavelength is predicted to have the most significant impact on the deviation from the theoretical design. In larger scale systems, the effects of thickness tolerance, imperfection sensitivity, and buckling wavelength would become negligible, allowing for practical experimental validation of the model. The optimized cylinder with a compliant core demonstrates promise in theory; however, in practice, manufacturing constraints limit the implementation of this design to relatively large, thin-walled structures composed of select materials with large radius to thickness ratios. Future work should explore large scale experimental validation of the design equations presented in this paper.

5. Conclusion

The optimization analysis provides tractable analytical equations, which can be used to successfully optimize the design of a cylinder with a compliant core. Design equations are presented for the optimal design of a thin-walled cylinder with compliant core, of given radius and specified materials, for a prescribed load bearing capacity in axial compression. The optimization is based on one parameter, the modulus ratio, which demonstrates the optimal configuration occurs when the cylinder with the compliant core is designed to fail in elastic buckling and material failure, simultaneously. Theoretically, the optimized cylinder with a compliant core demonstrates substantial improvement in axial load bearing capacity over comparable designs, including the equivalent hollow cylinder, the optimized hat-stiffened cylinder and the optimized sandwich designs. In practice, however, the optimal modulus ratio is difficult to achieve with commercially available engineering materials. Moreover, this design is only practical for implementation in structures requiring large radius to thickness ratios, where the weight to load bearing ratio is a critical element of design. Experimental validation of this model was challenging on a small scale due to manufacturing constraints. On a large scale the specified manufacturing difficulties would be significantly reduced or negligible. Therefore, it is feasible to consider future studies on implementation of this design in relatively large scale engineering structures, such as space shuttle fuel tanks, aircraft fuselages, and offshore oil platforms. The optimized shell with compliant core has enormous potential to be a competitive technology for a select group of existing engineering structures.

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Appendix A

A.1. Governing equations for honeycomb sandwich

According to Hutchinson, an auxiliary function is utilized to optimize a sandwich-filled cylindrical shell with $E_c/E = \alpha_o\eta^2$, corresponding to a foam core, according to Eq. (7). Hutchinson performed two analyses where optimization is based on t and t_c and η is given and where optimization is based on t, t_c, and η. Hutchinson determined the three distinct failure regions. The following optimization of a sandwich structure is based on a honeycomb core, governed by Eq. (6) where optimization is based on t, t_c and η. For each specified failure region only the corresponding constituents of Eq. (24) are active.

Eq. (25) is derived by setting the partial derivative of the auxiliary function in Eq. (24) with respect λ_2 equal to zero, which gives

$$N = 2t\sigma_y.$$

Similarly, Eq. (26) is derived by setting the partial derivative of the auxiliary function in Eq. (24) with respect λ_1 equal to zero, giving

$$\frac{t_c}{a} = \frac{N}{Et}\left[\frac{2}{\sqrt{1-\nu^2}} - \frac{8\eta^{-1}(t/a)^2}{3\alpha_o(1-\nu^2)}\right]^{-1}.$$

Substituting into this equation for a_1 and a_2 and reducing gives Eq. (26). The following solutions are applicable to the corresponding failure regions.

A.2. Overall buckling with face sheet yielding but no wrinkling

For this failure region only the components associated with yielding in Eq. (24) are active. Setting the partial derivative of Eq. (24) with respect to η equal to zero gives

$$\lambda_1 = \frac{-2\pi a^3 L\rho}{Ea_2 t^2\eta^{-2}}$$

Setting the partial derivative of Eq. (24) with respect to t_c equal to zero gives

$$\lambda_1 = \frac{-2\pi aL\rho\eta}{E(t/a)(a_1 - a_2\eta^{-1}(t/a))}$$

Combining the previous two equations gives Eq. (27a). Combining Eq. (27a) with the partial derivative of the auxiliary function, Eq. (24), with respect to λ_2 equal to zero gives Eq. (27b).

As before, Eq. (27c) is derived by setting the partial derivative of Eq. (24) with respect to λ_1 equal to zero. Combined, the previous equations give the constraint equations for the buckling with yielding failure region.

A.3. Overall buckling with face sheet wrinkling but no yielding

For this failure region only the components associated with wrinkling in Eq. (24) are active. Setting the partial derivative of Eq. (24) with respect to η equal to zero gives

$$\lambda_3 = \frac{-2\pi aLt_c - \lambda_1 Et_c(t/a)(a_2\eta^{-2}(t/a))}{4/3E\beta_o\alpha_o^{2/3}t\eta^{-1/3}}.$$

Setting the partial derivative of Eq. (24) with respect to t equal to zero gives

$$0 = 4\pi a l \rho + \lambda_1 \left(\frac{E t_c}{a} (a_1 - a_2 \eta^{-1} (2t/a)) \right) + \lambda_3 (2\beta_o \alpha_o^{2/3} \eta^{2/3})$$

Setting the partial derivative of Eq. (24) with respect to t_c equal to zero gives

$$0 = 2\pi a l \rho \eta + \lambda_1 \left(\frac{E t}{a} (a_1 - a_2 \eta^{-1} (t/a)) \right)$$

Combining the previous three equations gives the constraints given in Eq. (28a).

As before, Eq. (28b) is derived by setting the partial derivative of Eq. (24) with respect to λ_1 equal to zero. Combined, the previous equations give the constraint equations for the buckling with face sheet wrinkling failure region.

A.4. Overall buckling with face sheet yielding and wrinkling

For this failure region all of the components of Eq. (24) are active. Eq. (29a) is determined by setting the partial derivative of Eq. (24) with respect to λ_3 equal to zero, giving

$$N = 2\beta_o E \alpha_o^{2/3} t \eta^{2/3}.$$

Eq. (29b) is previously derived by setting the partial derivative of Eq. (24) with respect to λ_2 equal to zero. Similarly Eq. (29c) is obtained by setting the partial derivative of Eq. (24) with respect to λ_1 equal to zero as before. The previous three equations combined give the constraint equations for the region where buckling, yielding, and face sheet wrinkling occur simultaneously.

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